TUT 2: INTEGRATING FACTORS AND SEPARABLE EQUATIONS

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In this tutorial notes, I will give some examples to recap two basic methods to solve certain type ODEs, i.e. the use of integrating factors to solve first order linear ODEs and the method to solve separable equations. I will also include the homogeneous equations and Bernoulli equations.

In my tutorial notes, the informations in "Note" are important, and I suppose you would read it in detail; these in "Remark" give some expanding thinking or mathematical justification, which may motivate your future study. In addition, "example" means it has been given in the tutorial, and others are left "exercises".

1. The method of integrating factors

This method is usually used to solve an first order linear ODE with standard form

$$\frac{dy}{dx} + p(x)y = q(x). \tag{1}$$

If $p \equiv 0$, the solution is clear. Indeed, people introduce an **integrating factor** $\mu(x)$ to reduce (1) to an ODE without p-term. Considering

$$\begin{aligned} \frac{d}{dx}(\mu(x)y(x)) &= \mu y' + \mu' y = \mu[-p(x)y + q(x)] + \mu' y \\ &= \big[\mu' - \mu p(x)\big]y + q(x)\mu, \end{aligned} \tag{2}$$

which can be reduced to solve

$$\begin{cases} \mu' - \mu p(x) = 0, \quad \mu > 0; \\ (\mu y)' = q(x)\mu(x). \end{cases} \tag{3}$$

you can solve the first equation, and then the second, which gives the solution y.

Note 1. *The condition, $\mu > 0$, is necessary, which ensures that solving equation* (3) *really solves* (1) *(check the Lecture notes for detailed mathematical justification).*

In the following, let's try this method by some examples. If the equation is not in standard form, we first need to rewrite it.

Example 1. *Find the general solution of*

$$(1 + x^2)y' + 4xy = (1 + x^2)^{-2}.$$

Solution.

Step 1. (Rewrite the equation in standard form) The above equation can be written in the standard form,

$$y' + \frac{4x}{1 + x^2}y = (1 + x^2)^{-3}.$$

Step 2. (Introduce the integrating factor μ) Let μ be a function of x, and considering

$$\begin{aligned}(\mu y)' = \mu' y + \mu y' &= \mu' y + \left(-\frac{4x}{1+x^2}y + \frac{1}{(1+x^2)^3}\right)\mu \\ &= \left(\mu' - \frac{4x}{1+x^2}\mu\right) y + \frac{\mu}{(1+x^2)^3}.\end{aligned} \tag{4}$$

Step 3. (Determine μ) Suppose that μ satisfies

$$\mu' - \frac{4x}{1+x^2}\mu = 0.$$

By solving it, one can take

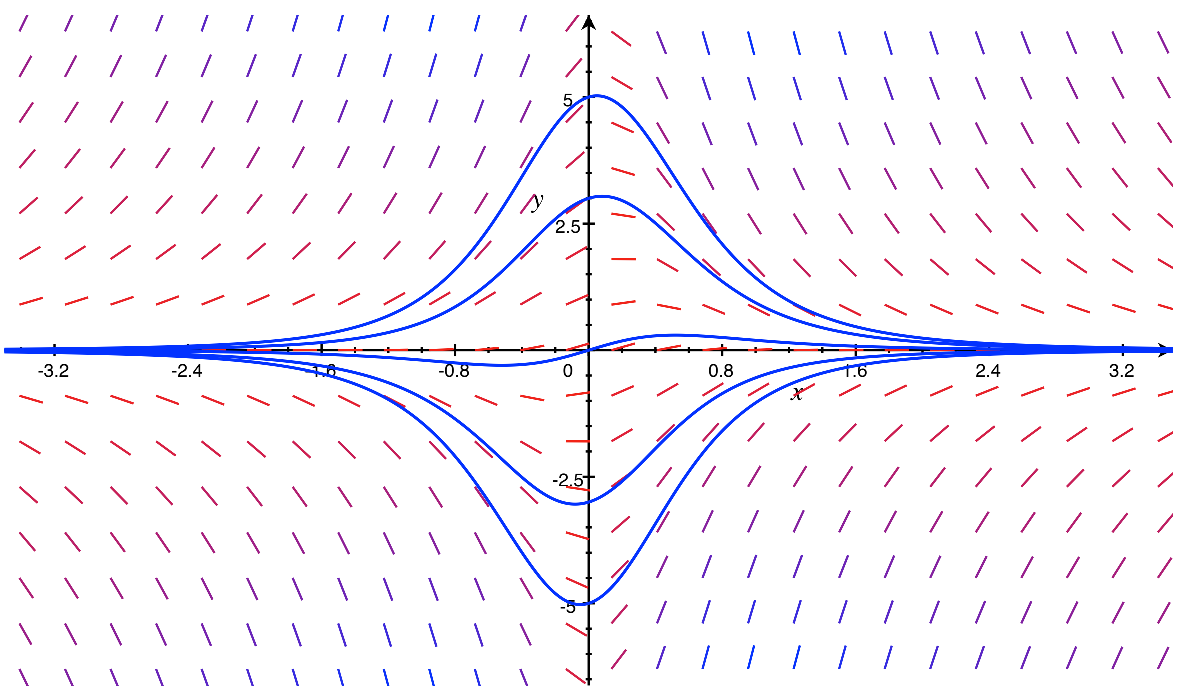
$$\mu = (1+x^2)^2 > 0.$$

Step 4. (Back to Step 2 and solve y) Back to (4),

$$\frac{d}{dx}\big[(1+x^2)^2 y(x)\big] = \frac{1}{(1+x)^2},$$

hence for some real number c,

$$y = (1+x^2)^{-2}(\arctan x + c).$$



Exercise 2. *Solve the initial value problem (IVP)*

$$xy' + (x+1)y = 2xe^{-x} \quad \textit{with} \quad y(1) = a \qquad (x > 0). \tag{5}$$

And then describe the behaviour of solution near 0 *corresponding to different initial value* a.

Solution. Rewriting the equation as

$$y' + \frac{x+1}{x}y = 2e^{-x} \tag{6}$$

for $x > 0$. Let μ be a function of x, and considering

$$\begin{aligned} \frac{d}{dx}(\mu y) &= \mu' y + \mu y' = \mu' y + \mu\left(-\frac{x+1}{x}y + 2e^{-x}\right) \\ &= \left(\mu' - \frac{x+1}{x}\mu\right) y + 2\mu e^{-x}. \end{aligned} \tag{7}$$

Choosing $\mu > 0$ such that

$$\mu' - \frac{x+1}{x}\mu = 0. \tag{8}$$

Since for $x > 0$,

$$(\ln \mu)' = \frac{\mu'}{\mu} = \frac{x+1}{x} = 1 + \frac{1}{x} = (x + \ln x)'.$$

We can choose

$$\mu(x) = e^{x+\ln x} = xe^x > 0,$$

which solves (8). Back to (7), we have

$$\frac{d}{dx}(xe^x y) = 2x.$$

Integrating implies when $x > 0$,

$$y(x) = x^{-1}e^{-x}(x^2 + c)$$

for some constant c. By the initial condition,

$$y(1) = e^{-1}(1 + c) = a,$$

then

$$c = ae - 1.$$

Hence, the solution of the IVP is

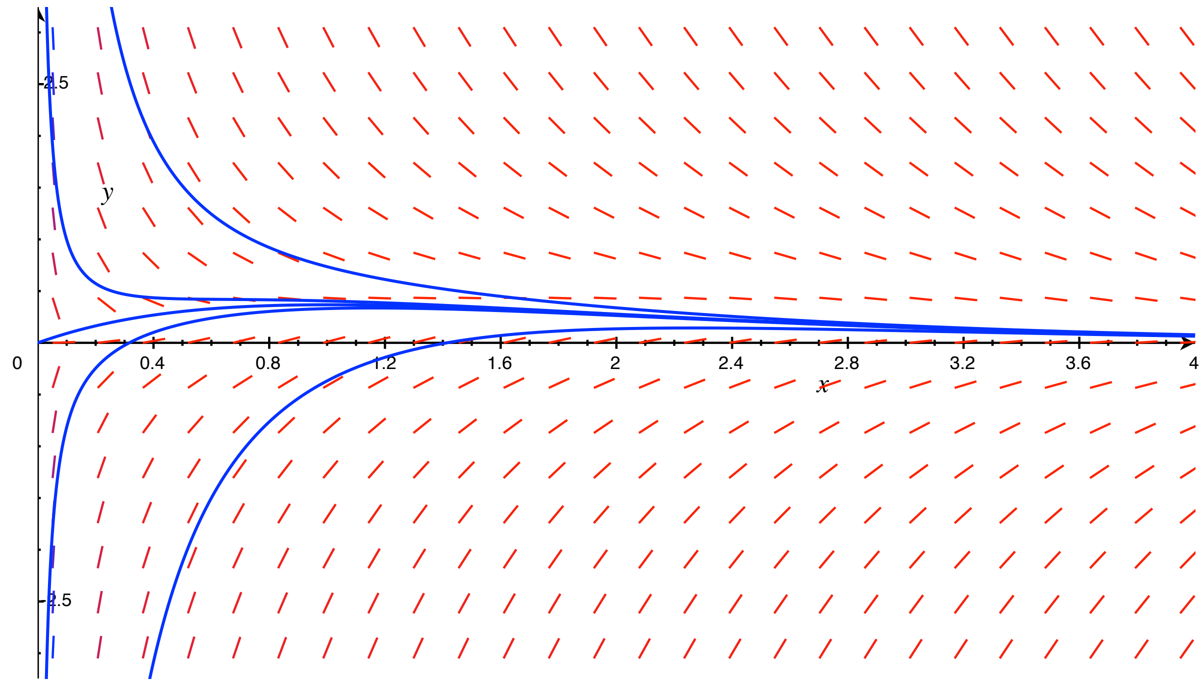
$$y(x) = x^{-1}e^{-x}(x^2 + ae - 1).$$

By the above representation of the solution,

Table 1. Behaviour near 0

a	$a > e^{-1}$	$a = e^{-1}$	$a < e^{-1}$
$ae - 1$	$ae - 1 > 0$	$ae - 1 = 0$	$ae - 1 < 0$
as $x \to 0^+$	$y(x) \to +\infty$	$y(x) \to 0$	$y(x) \to -\infty$

We can see from the above table that the behaviour of the solution depends on the initial data a, and when a cross e^{-1}, the transition from one type of behaviour to another occurs.



2. Separable equations

The standard form of a separable ODE is

$$f(y)\frac{dy}{dx} = g(x) \quad \text{or} \quad f(y)\,dy = g(x)\,dx, \tag{9}$$

which could be nonlinear. Denote

$$F(y) = \int f(y)\,dy.$$

Hence integrating implies

$$F(y) = \int g(x)\,dx + c =: G(x) + c$$

for some real number c. Note that the solution may be in implicit form.

Example 3. *Find the general solution of*

$$\frac{dy}{dx} = xy(4-y). \tag{10}$$

Solution. Consider where $y \neq 0, 4$, we can rewrite the equation as

$$\frac{1}{y(4-y)}\frac{dy}{dx} = x. \tag{11}$$

Since

$$\frac{1}{y(4-y)} = \frac{A}{y} + \frac{B}{4-y} \quad \left(= \frac{A(4-y)+By}{y(4-y)} = \frac{(B-A)y+4A}{y(4-y)} \right)$$

with

$$A = B = 1/4.$$

Back to (10), chain rule implies

$$\frac{1}{4}\left(\frac{1}{y}+\frac{1}{4-y}\right)\frac{dy}{dx}=\frac{1}{4}\frac{d}{dx}\left(\ln|y|-\ln|4-y|\right)=x$$

Integrating gives

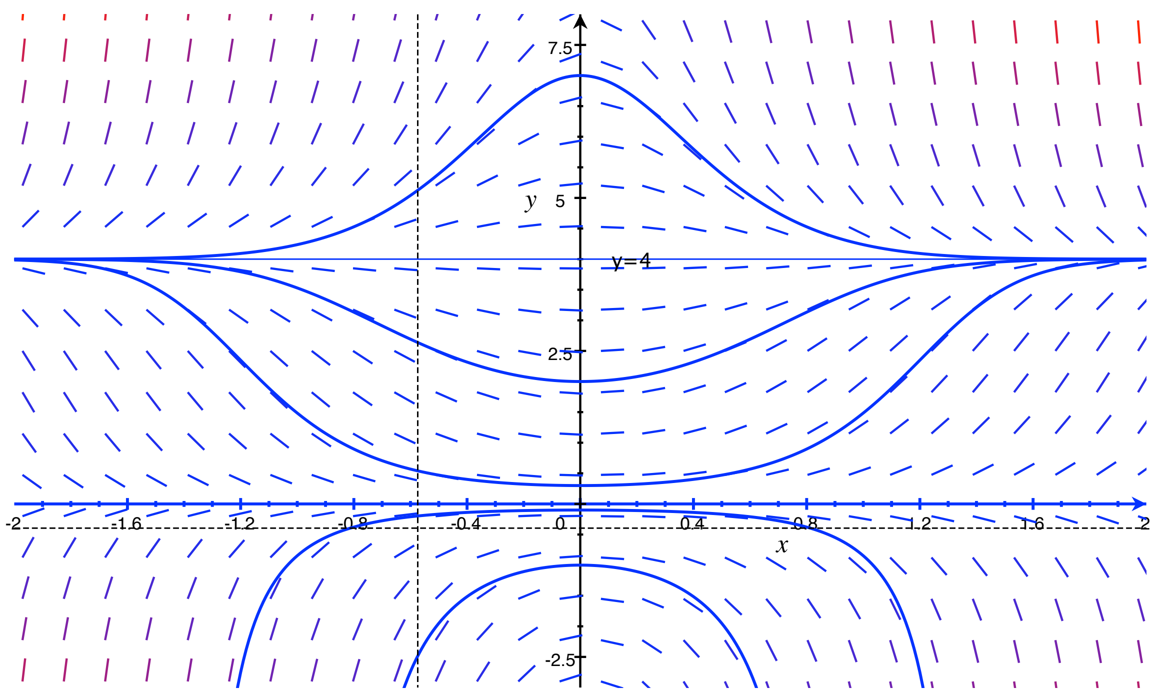
$$y=c(4-y)e^{2x^2}$$

for some constant $c \in \mathbb{R}$. And $y \equiv 0, 4$ are also solutions, but $y \equiv 4$ is not contained in the expression. The general solution actually is

$$c_2 y=c_1(4-y)e^{2x^2}$$

for some real numbers c_1, c_2.

You may ask how about the solution passing through 0 or 4, please see Remark 1.



Except separable equations like above, there are some special types of ODEs such as homogeneous ODEs and Bernoulli equations, which can be reduced to separable ODEs by transformations.

Example 4 (Homogeneous equation). *Find the general solution for the nonlinear, non-seperable ODE*

$$\frac{dy}{dx}=\frac{x^2+3y^2}{2xy}=\frac{1+3y^2/x^2}{2y/x} \quad \textit{for} \quad x\neq 0. \tag{12}$$

Solution.

Step 1. Introduce a new independent variable $z=y/x$, i.e. $y=zx$. Then

$$\frac{dy}{dx}=\frac{dz}{dx}x+z=\frac{1+3(y/x)^2}{2y/x}=\frac{1+3z^2}{2z}.$$

Step 2. (Reduce to the eq of z) As a equation of z,

$$x\frac{dz}{dx}=\frac{1+3z^2}{2z}-z=\frac{1+z^2}{2z},$$

which is a separable equation, and can be rewritten as

$$\frac{2z}{1+z^2}\,dz = \frac{1}{x}\,dx$$

Step 3. (Solve the separable eq of z) Integrating gives

$$z^2 + 1 = cx$$

for some real number c.

Step 4. (Go back to the original dependent variable) And by $z = y/x$,

$$y^2 + x^2 = cx^3, \quad x \neq 0. \tag{13}$$

Remark 1. *This equation is not valid at $y = 0$. But firstly we know $y \equiv 0$ is not solution for any interval with nonzero length. And suppose we admit the solution is continuous on its validity domain, from the expression of solution, if $0 = y(x^*) = \lim_{x\to x^*} y(x) = c(x^*)^3 - (x^*)^2$ for some x^*, then $x^* = 0$, which is impossible since $x \neq 0$, or $x^* = 1/c$ which is contained in the above expression of the solution. This remark gives a mathematical justification that* (13) *is really the general solution of* (12).

On this basis, I give the following suggestions.

Note 2. *When you have a equation, first consider the valid domain both for x and y, and find the solution. When the invalid domain is a point, omit it. When the invalid domain is a line, then first check whether it is a solution, if it is, then check whether it is contained in the solution you got above or not, while the answer is no, you have to rewrite the solution; if it is not, omit it.*

Remark 2. *Consider the equation with initial data $y(1) = 1$, then $c = 2$ and the solution is $y^2 = 2x^3 - x^2$, which is negative for $x < 1/2, x \neq 0$. This particularly, means that the solution does not exist for $x < 1/2$. Moreover you can see the domain of existence depends on the value c which is determined by the initial data.*

Note 3. *The above method can be used in general homogeneous equation*

$$\frac{dy}{dx} = f(y/x).$$

Introduce $\underline{z = y/x}$, then

$$\frac{dy}{dx} = \frac{dz}{dx}x + z = f(z) \quad \textit{and then} \quad \frac{1}{f(z)-z}\frac{dz}{dx} = \frac{1}{x}.$$

For homogeneous here, we mean by scaling $(X, Y) = (\lambda x, \lambda y)$ for $\lambda \neq 0$, the equation for Y with independent variable X keeps to be the same. And f is called a homogeneous function with degree 0. I mention quasi-homogeneous equation as an expansion for interested students.

Exercise 5. *Find the general solution for*

$$\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{y/x-4}{1-y/x}. \tag{14}$$

Solution: Consider where $x \neq 0$. Introduce a new dependent variable $z := y/x$, i.e. $y = zx$. Then

$$\frac{dy}{dx} = \frac{dz}{dx}x + z = \frac{z-4}{1-z}.$$

Rewriting the above equation to

$$x\frac{dz}{dx} = \frac{z-4}{1-z} - z = \frac{z-4-z(1-z)}{1-z} = \frac{z^2-4}{1-z},$$

which is a separable equation. Furthermore,

$$\frac{1-z}{z^2-4}\,dz = \frac{1}{x}\,dx.$$

Since

$$\begin{aligned}\frac{1-z}{z^2-4} &= (1-z)\left(\frac{A}{z-2} + \frac{B}{z+2}\right) \qquad \big(\text{for} \quad A = -B = 1/4\big)\\ &= \frac{1}{4}\left(\frac{-(z-2)-1}{z-2} - \frac{3-(z+2)}{z+2}\right)\\ &= \frac{1}{4}\left(-\frac{1}{z-2} - \frac{3}{z+2}\right) = -\frac{1}{4}\frac{d}{dz}\ln\big(|z-2||z+2|^3\big).\end{aligned} \tag{15}$$

Hence

$$(z-2)(z+2)^3 = cx^{-4}$$

for some real number c. Since $z = y/x$, then

$$(y-2x)(y+2x)^3 = c.$$

Example 6 (Bernoulli equations). *Let n be a real number and $n \neq 0, 1$, find the general solution for the first order nonlinear, non-separable ODE*

$$\frac{dy}{dx} + p(x)y = q(x)y^n. \tag{16}$$

Solution. It is not first order ODE, so we can not use integrator factor to solve it directly. But it seems almost like an linear ODE. Indeed, considering where $y \neq 0$ and dividing the equation by y^n, we have

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x),$$

where the right hand side is independent of y. We are getting closer to an linear ODE! And

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{d}{dx}y^{1-n}.$$

Hence we introduce a new depend variable $\underline{z = y^{1-n}}$, and really have an linear ODE

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x).$$

Then introduce the integrating factor $\mu(x)$ to solve this linear ODE as we do at the beginning of the tutorial. Then y is given by

$$y(x) = z(x)^{1/(1-n)}.$$

The last thing is to verify $y = 0$ is also contained in the expression.